

# UNIVERSALITY THEOREMS FOR INSCRIBED POLYTOPES AND DELAUNAY TRIANGULATIONS

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**ABSTRACT.** We prove that every primary basic semialgebraic set is homotopy equivalent to the set of inscribed realizations (up to Möbius transformation) of a polytope. If the semialgebraic set is moreover open, then, in addition, we prove that (up to homotopy) it is a retract of the realization space of some inscribed neighborly (and simplicial) polytope. We also show that all algebraic extensions of  $\mathbb{Q}$  are needed to coordinatize inscribed polytopes. These statements show that inscribed polytopes exhibit the Mnëv universality phenomenon.

Via stereographic projections, these theorems have a direct translation to universality theorems for Delaunay subdivisions. In particular, our results imply that the realizability problem for Delaunay triangulations is polynomially equivalent to the existential theory of the reals.

## 1. INTRODUCTION

The Delaunay subdivision of a set of points in  $\mathbb{R}^d$  plays a central role in computational geometry [Ede06]. A few applications are: nearest-neighbor search, pattern matching, clustering, and mesh generation. Via stereographic projection, Delaunay subdivisions can be lifted to *inscribed polytopes* [Bro79]—those with all vertices on the unit sphere—in one dimension higher, so that Delaunay triangulations lift to simplicial inscribed polytopes. The study of inscribed polytopes, and in particular the problem of deciding whether a polytope admits an inscribed realization, is a classical subject [Ste32][Ste28][Riv94] in which many fundamental questions are still open [GZ11].

In this paper, we are interested in *realization spaces* of a *fixed combinatorial type* of Delaunay subdivision/inscribed polytope. For a configuration  $A$  of  $n$  points in  $\mathbb{R}^d$ , which we assume to be labeled by  $[n] = \{1, \dots, n\}$ , the cells of its Delaunay subdivision are represented by a family  $T$  of subsets of  $[n]$ . The *realization space*  $\mathcal{R}_{\text{del}}(T)$  is a parametrization of the set of all configurations of  $n$  labeled points whose Delaunay triangulation has the combinatorial structure of  $T$  (as a polytopal complex with vertex set  $[n]$ ).

Analogously,  $\mathcal{R}_{\text{ins}}(P)$ , the realization space of an inscribed polytope  $P$ , is a parametrization of configurations of  $n$  points in the unit sphere whose convex hull has the same face lattice as  $P$ .

In dimension 2, results of [Riv94] imply that both of these realization spaces are homeomorphic to a polytope (that depends on  $T$  only). This completely determines the topological structure of both realization spaces. For example, they are always *connected* and *contractible*.

**1.1. Universality for Delaunay subdivisions.** Our main results show that, in higher dimensions, the situation is completely different: the realization space of a  $d$ -dimensional Delaunay subdivision (or triangulation) can be arbitrarily complicated.

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**Theorem 3.1.** *For every primary basic semi-algebraic set there is a Delaunay subdivision and an inscribed polytope whose realization space is homotopy equivalent to  $S$ .*

Said differently, realization spaces of inscribed polytopes exhibit the *topological universality* in the sense of Mnëv [Mnë88]. We also show that these realization spaces exhibit *algebraic universality* (also a notion from [Mnë88]).

**Corollary 3.5.** *For every finite field extension  $F/\mathbb{Q}$  of the rationals, there is a realizable Delaunay subdivision (equivalently, an inscribed polytope) that cannot be realized with coordinates in  $F$ .*

**1.2. Universality for Delaunay triangulations.** The subdivisions constructed in proof of Theorem 3.1 are far from being triangulations. To insist on triangulations and simplicial polytopes requires different tools. We adapt a recent proof of the Universality Theorem for simplicial polytopes [AP14] to obtain a weak universality theorem for Delaunay triangulations.

**Theorem 4.15.** *For every open primary basic semi-algebraic set  $S$  there is a (neighborly) Delaunay triangulation and an inscribed simplicial (neighborly) polytope, such that  $S$  is a retract of their realization spaces, up to homotopy equivalence.*

**1.3. Complexity.** The complete statements of these theorems provide linear bounds for the number points of the triangulation and the dimension in terms of the *arithmetic complexity* of the corresponding semi-algebraic sets. Moreover,  $S$  is non-empty if and only if  $\mathcal{R}_{\text{del}}(T)$  is. Such a triangulation can be computed from  $S$  in polynomial time, which shows that deciding whether a Delaunay triangulation is realizable is hard. Indeed, the proof of Theorem 4.15 shows that deciding realizability of Delaunay triangulations is as hard as deciding realizability of rank 3 oriented matroids [Mnë88][Sho91].

**Corollary 4.16.** *The realizability problem for Delaunay triangulations and inscribed simplicial polytopes is polynomially equivalent to the existential theory of the reals (ETR). In particular, it is NP-hard.*

Another consequence of this effective bound is that the number of connected components of the realization space of a  $d$ -dimensional Delaunay triangulation can be exponential in  $d$ .

**Corollary 4.17.** *For every  $m \geq 1$  there exist configurations of  $O(m)$  points in general position in  $\mathbb{R}^{O(m)}$  whose realization spaces as Delaunay triangulations have at least  $2^m$  connected components.*

Our smallest example of a triangulation with disconnected realization space is in  $\mathbb{R}^{25}$ , which leaves open the existence of these configurations in  $\mathbb{R}^d$  for each  $3 \leq d \leq 24$ .

**Corollary 4.18.** *There is a 25-dimensional configuration of 30 points whose Delaunay triangulation has a disconnected realization space.*

**1.4. Context and related work.** The realization spaces of 3-dimensional inscribed polyhedra are well understood. On the other hand, there is a rich theory of the “wildness” of realization spaces of higher-dimensional polyhedra can be quite wild. Here is the background and connection with our results.

**1.4.1. Dimension 2 and inscribable polyhedra.** Theorems 3.1 and 4.15 and their corollaries should be contrasted with fundamental results of Rivin [Riv94][Riv96][Riv03] that connect 2-dimensional Delaunay subdivisions with metric properties of hyperbolic 3-dimensional polyhedra.

Rivin’s work in particular entailed that: (1) whether a (combinatorial) planar graph has a drawing as a Delaunay triangulation can be tested in polynomial time; (2) that the realization space of a planar Delaunay triangulation is homeomorphic to a polyhedron of so-called *angle structures* (see [FG11] for an elementary introduction to the method), and, in particular, connected.

In the language of polyhedra, (1) says that whether a graph is the 1-skeleton of an inscribable polyhedron is efficiently checkable; and (2) says that the set of inscribed realizations is convex (and in particular contractible) in the parameterization by dihedral angles.

The question of whether every polyhedron is inscribable had been first raised by Steiner in 1832 [Ste32], with the first negative examples given by Steinitz in 1928 [Ste28]. This makes such a sharp characterization of the inscribable types and their realization spaces a surprising breakthrough. In contrast, Theorems 3.1 and 4.15 suggest that a polynomial time characterization for all dimensions is, under standard conjectures, not possible.

**1.4.2. Higher dimensions and universality.** A general principle in the theory of realization spaces for (semi-)algebraically defined objects is succinctly put in [Vak06]: “Unless there is some a priori reason otherwise, the deformation space may be as bad as possible.”

Underlying a large number of these kinds of phenomena is a paradigmatic result of Mnëv. The Universality Theorem states that for every [open] primary basic semi-algebraic set there is a [uniform] oriented matroid of rank 3 whose realization space is stably equivalent to it. (The survey [RG98] provides an accessible presentation of this and related results and their proofs, and a more computationally oriented approach can be found in [Sho91].)

The Universality Theorem in particular entails a negative answer to Ringel’s 1956 *isotopy problem*, which asked whether, given two point configurations  $A_0$  and  $A_1$  with the same oriented matroid (order type), is it always possible to find a continuous path of point configurations  $\{A_t\}_{0 \leq t \leq 1}$  with the same oriented matroid? (This weaker result also follows via examples from [JMLSW89] [Ric96] [Suv88] [Tsu13] [Ver88] [Whi89].) Actually, the Universality Theorem shows that there are oriented matroids that have realization spaces with arbitrarily many components.

Another straightforward consequence of the Universality Theorem is that determining realizability of oriented matroids is polynomially equivalent to the *existential theory of the reals*, and in particular NP-hard [Mnë88][Sho91].

Via a reduction given in [Mnë88], realization spaces of polytopes also exhibit universality: for every semi-algebraic variety  $S \subset \mathbb{R}^s$ , there is a polytope  $P$  whose realization space is stably equivalent to  $S$ . Here, the realization space of a polytope  $P$  is the set of point configurations whose convex hull is combinatorially equivalent to  $P$ . In principle, this polytope might be of a very high dimension. However, Richter-Gebert [RG96] made a breakthrough when he proved that there is universality already in realization spaces of 4-dimensional polytopes. Again, there is a contrast with 3-polytopes, which have contractible realization spaces (see, e.g., [RG96, Part IV]).

When the semi-algebraic sets are open, one can furthermore require the polytopes to be simplicial (and even neighborly), although only in arbitrarily high dimensions [AP14][Mnë88]. Universality for simplicial polytopes in fixed dimension remains wide open.

However, the existence of Delaunay triangulations with a disconnected realization space is not a direct consequence of the results of Richter-Gebert and Mnëv. Indeed, even if  $P$  is a polytope with a disconnected realization space, it could be that the variety  $\mathcal{S}$  that certifies that all the vertices lie on the unit sphere does not intersect with all its connected components—or any of them. Hence, to present a Delaunay triangulation with a disconnected realization space, one has to show that  $\mathcal{S}$  hits at least two of these connected components or that the intersection of  $\mathcal{S}$  with a connected component is disconnected.

On the other hand, some universality phenomena from the theory of general polytopes are already known to carry over to the case of inscribed polytopes; for instance, there are infinitely many projectively unique inscribed polytopes even in *bounded* dimension, and every inscribed polytope is the face of some projectively unique inscribed polytope, cf. [AZ14].

**1.5. Open problem: universality in fixed dimension.** Theorems 3.1 and 4.15 are the first step towards a universality theory for Delaunay triangulations and leave several open questions. First of all, a strong version of Theorem 4.15 should state homotopy equivalence between the realization space of the Delaunay triangulation and the semi-algebraic set.

The main challenge is to prove a Universality Theorem for Delaunay Triangulations in fixed dimension. Recall that polytopes present universality already in dimension 4 (for simplicial polytopes this is also conjectured). Since the results here run more or less in parallel with the development of the theory for polytopes, the strongest conjecture we can make is:

**Conjecture 1.1.** *For every [open] primary basic semi-algebraic set  $S$  defined over  $\mathbb{Z}$  there is a 3-dimensional Delaunay [triangulation] subdivision whose realization space is homotopy equivalent to  $S$ .*

Since connected sums, the main ingredient of Richter-Gebert's proof of the Universality Theorem for 4-polytopes [RG96], do not behave well with respect to inscribability, it seems that a new set of tools will be needed to prove our conjecture.

**1.6. Reading guide.** The rest of this paper is organized as follows: Section 2 introduces some necessary notation. Section 3 is devoted to the Universality Theorem for inscribed polytopes and Delaunay subdivisions. Simplicial polytopes and triangulations require different tools, and are studied in Section 4.

## 2. PRELIMINARIES

**2.1. Notation.** For a quick reference for oriented matroids, polytopes and Delaunay triangulations, we refer to the chapters [RGZ97], [HRGZ97] and [For97] of the handbook [GO04], respectively. Our notation coincides mostly with theirs.

Let  $V$  be a configuration of  $n$  vectors in  $\mathbb{R}^r$ , which are labeled by elements in  $[n] = \{1, \dots, n\}$ . Consider the map  $\chi^V : [n]^r \mapsto \{0, 1, -1\}$  that for each tuple  $(i_1, \dots, i_r)$  assigns the sign

$$\chi^V(i_1, \dots, i_r) = \text{sign det}(v_{i_1}, \dots, v_{i_r}).$$

The map  $\chi^V$  is called the *chirotope* of  $V$  and determines its *oriented matroid*, which has *rank*  $r$  (see [BLS<sup>+</sup>93] for a comprehensive introduction to oriented matroids).

Now, let  $A$  be a configuration of  $n$  points in  $\mathbb{R}^d$ . The *homogenization* of  $A$  is a vector configuration  $\text{hom}(A) = \{\bar{a}_1, \dots, \bar{a}_n\} \subset \mathbb{R}^{d+1}$  obtained by appending 1 as the last coordinate of the points of  $A$ :  $\bar{a}_i = (a_i, 1)$ . The *oriented matroid* of  $A$  is defined to be the oriented matroid of its homogenization  $\text{hom}(A)$ .

The oriented matroid of a point configuration is always *acyclic*. A point configuration  $A$  is in *general position* if no  $d + 1$  points of  $A$  lie in a common hyperplane, and then it defines a *uniform* matroid.

The convex hull of  $A$  is a *polytope*  $P = \text{conv}(A) \subset \mathbb{R}^d$  and the intersection of  $P$  with a supporting hyperplane is a *face* of  $P$ . Faces of dimensions 0 and  $d - 1$  are called *vertices* and *facets*, respectively. A point configuration  $A$  is in *convex position* if it coincides with  $\text{vert}(P)$ , the set of vertices of  $P = \text{conv}(A)$ . If  $A$  is in convex position, each face  $F$  of  $P$  can be identified with the set of labels  $\{i \in [n] \mid a_i \in F\}$ . The *face lattice* of  $P$  is then a poset of subsets of  $[n]$ . In this context, two vertex-labeled polytopes are *combinatorially equivalent*, denoted  $P \simeq Q$ , if their face lattices coincide. We call  $P$  *inscribed* if all its vertices lie in the unit sphere  $\mathbb{S}^{d-1}$ , and *inscribable* if it is combinatorially equivalent to an inscribed polytope.

The face lattice of a polytope  $P$  coincides with that of the *convex cone* obtained as the positive hull of its homogenization  $\text{pos}(\text{hom}(P)) := \{\sum \lambda_i x_i \mid \lambda_i \geq 0, x_i \in \text{hom}(P)\}$ .

The oriented matroid of a polytope  $P$  is *rigid* if its face lattice determines the oriented matroid of its set of vertices (see [Zie95, Section 6.6]). In the language of the next section, a polytope  $P$  is rigid if and only if  $\mathcal{R}_{\text{om}}(\text{vert}(P)) = \mathcal{R}_{\text{pol}}(P)$ .

A *subdivision* of a point configuration  $A$  is a collection  $\mathcal{T}$  of polytopes with vertices in  $A$ , which we call *cells*, that cover the convex hull of  $A$  and such that any pair of polytopes of  $\mathcal{T}$  intersect in a common face. A *triangulation* of  $A$  is a subdivision where all the cells are simplices. Again, a subdivision of  $A$  can be identified with a poset of subsets of  $[n]$ . Two subdivisions  $\mathcal{T}$  and  $\mathcal{T}'$  of two labeled configurations  $A$  and  $A'$  are *combinatorially equivalent*, denoted by  $\mathcal{T} \simeq \mathcal{T}'$ , if their respective posets coincide.

The *Delaunay subdivision*  $\mathcal{D}(A)$  of a point configuration  $A \subset \mathbb{R}^d$  is the subdivision that consists of all cells defined by the *empty circumsphere condition*:  $S \in \mathcal{D}(A)$  if and only if there exists a  $(d-1)$ -sphere that passes through all the vertices of  $S$  and all other points of  $A$  lie outside this sphere. If  $A$  is in general position and no  $d+2$  points of  $A$  lie on a common sphere, then the empty circumsphere condition always defines a simplex of  $A$ , and hence the Delaunay subdivision is a triangulation, the *Delaunay triangulation* of  $A$ .

We denote *homeomorphic* sets  $S$  and  $T$  by  $S \cong T$  and *homotopic* sets by  $S \sim T$  (see [Mun75, Section 58] for definitions). We also recall that a continuous map  $f : S \rightarrow T$  is a *retraction* of  $S$  onto  $T$  if there is a continuous map  $g : T \rightarrow S$  such that  $f \circ g = \text{id}$ . If a retraction exists, then  $T$  is a *retract* of  $S$ . If moreover  $g \circ f$  is homotopic to the identity, then  $T$  is a *deformation retract* of  $S$ , and  $T \sim S$ .

**2.2. Realization spaces.** We will work with the following *realization spaces*. Observe that for oriented matroids and polytopes, we work with the acyclic vector configurations arising from homogenization. (This approach is convenient for technical reasons, and used often, for example in [BLS<sup>+</sup>93].) We also identify a  $d$ -dimensional configuration of  $n$  points or vectors with the corresponding tuple in  $\mathbb{R}^{d \times n}$  containing the coordinates, ordered according to their labels.

- The *realization space of an oriented matroid*  $M$  (of rank  $d+1$  with  $n$  elements), that we denote  $\mathcal{R}_{\text{om}}(M) \subset \mathbb{R}^{(d+1) \times n}$ , is the set of vector configurations that realize  $M$ , up to linear transformation:

$$\mathcal{R}_{\text{om}}(M) = \left\{ V \in \mathbb{R}^{(d+1) \times n} \mid V \text{ realizes } M \right\} / \text{GL}(\mathbb{R}^{d+1}).$$

- The *realization space of a polytope*  $P$  (with  $n$  vertices in  $\mathbb{R}^d$ ), that we denote  $\mathcal{R}_{\text{pol}}(P) \subset \mathbb{R}^{(d+1) \times n}$ , is the set of acyclic configurations whose positive span is combinatorially equivalent to the cone over  $P$ , up to linear transformation:

$$\mathcal{R}_{\text{pol}}(P) = \left\{ V \in \mathbb{R}^{(d+1) \times n} \mid \text{pos}(V) \simeq P \right\} / \text{GL}(\mathbb{R}^{d+1}).$$

- The *realization space of an inscribed polytope*  $P$  (with  $n$  vertices in  $\mathbb{R}^d$ ), that we denote  $\mathcal{R}_{\text{ins}}(P) \subset (\mathbb{S}^{d-1})^n \subset \mathbb{R}^{d \times n}$ , is the set of inscribed point configurations whose convex hull is combinatorially equivalent to  $P$ , up to Möbius transformation:

$$\mathcal{R}_{\text{ins}}(P) = \left\{ A \in (\mathbb{S}^{d-1})^n \mid \text{conv}(A) \simeq P \right\} / \text{Möb}(\mathbb{S}^{d-1}).$$

- The *realization space of a Delaunay subdivision*  $T$  (of  $n$  points in  $\mathbb{R}^d$ ), that we denote  $\mathcal{R}_{\text{del}}(T) \subset \mathbb{R}^{d \times n}$ , is the set of point configurations whose Delaunay triangulation is combinatorially equivalent to  $T$ , up to similarity:

$$\mathcal{R}_{\text{del}}(T) = \left\{ A \in \mathbb{R}^{dn} \mid \mathcal{D}(A) \simeq T \right\} / \text{Sim}(\mathbb{R}^d).$$



For a given point configuration  $A$ , we abuse notation and use  $\mathcal{R}_{\text{om}}(A)$ ,  $\mathcal{R}_{\text{pol}}(A)$ ,  $\mathcal{R}_{\text{ins}}(A)$ , and  $\mathcal{R}_{\text{del}}(A)$  to denote the realization spaces  $\mathcal{R}_{\text{om}}(\chi^A)$ ,  $\mathcal{R}_{\text{pol}}(\text{conv}(A))$ ,  $\mathcal{R}_{\text{ins}}(\text{conv}(A))$  and  $\mathcal{R}_{\text{del}}(\mathcal{D}(A))$ , respectively.

**Remark 2.1.** A number of alternative definitions are possible. For example, by factoring different transformation groups or by considering non-homogenized configurations. Most of these definitions are actually homotopy-equivalent, as we discuss below, and hence our results hold anyway. These definitions are natural because the groups preserve spheres.

We also leave out the combinatorics of the boundary for Delaunay subdivisions, which amounts to take also into account the empty spheres that go through the “point at infinity”. Although these two definitions are not necessarily homotopy-equivalent, again, our results hold for both kinds of definition, see also Remark 4.3.

It is sometimes useful to commute between different realization spaces; we state the straightforward lemmata, without detailed proof, here:

**Lemma 2.2** (Realization spaces of matroids). *Let  $M$  be an acyclic oriented matroid. Then the following three spaces are homotopy equivalent:*

◦ *The realization space of homogeneous configurations, modulo linear transformations:*

$$(1) \quad \mathcal{R}_{\text{om}}(M) = \left\{ V \in \mathbb{R}^{(d+1) \times n} \mid V \text{ realizes } M \right\} / \text{GL}(\mathbb{R}^{d+1})$$

◦ *The realization space of affine configurations, modulo admissible projective transformations:*

$$(2) \quad \mathcal{R}_{\text{om}}^{\text{proj}}(M) = \left\{ V \in \mathbb{R}^{d \times n} \mid V \text{ realizes } M \right\} / \text{PGL}(\mathbb{R}^d)$$

◦ *The realization space of affine configurations, modulo affine transformations:*

$$(3) \quad \mathcal{R}_{\text{om}}^{\text{aff}}(M) = \left\{ A \in \mathbb{R}^{d \times n} \mid A \text{ realizes } M \right\} / \text{Aff}(\mathbb{R}^d)$$

*Proof (sketch).* To see (1)~(2), consider the map  $\mathcal{R}_{\text{om}}^{\text{lin}}(M) \rightarrow \mathcal{R}_{\text{om}}^{\text{proj}}(M)$  that sends a vector configuration  $V$  to the intersection of its positive span with a hyperplane that intersects every positive ray spanned by  $V$ . The map is well defined because two point configurations arising from different hyperplanes are related by an admissible projective transformation, and all linear transformations of  $V$  also induce admissible projective transformations of  $A$ . The homogenization map provides a section, and the fibers are easily seen to be contractible.

For (2)~(3), observe that each fiber of the quotient map  $\mathcal{R}_{\text{om}}^{\text{aff}}(M) \rightarrow \mathcal{R}_{\text{om}}^{\text{proj}}(M)$  is homeomorphic to the set of admissible projective transformations up to affine transformation. That is, the set of “hyperplanes at infinity” that do not cut  $\text{conv}(A)$ . This, in turn, is homeomorphic to a polytope, the polar polytope of  $\text{conv}(A)$ , which depends continuously on  $A$ . Hence, a continuous section can be defined by selecting its barycenter.  $\square$

Similarly, we have the following lemma for inscribed polytopes:

**Lemma 2.3** (Realization spaces of inscribed polytopes). *Let  $P$  be an inscribed polytope in  $\mathbb{R}^d$ . Then the following three spaces are homotopy equivalent:*

◦ *The realization space of all inscribed polytopes combinatorially equivalent to it, modulo Möbius transformations:*

$$(4) \quad \mathcal{R}_{\text{ins}}(P) = \left\{ A \in (\mathbb{S}^{d-1})^n \mid \text{conv}(A) \simeq P \right\} / \text{Möb}(\mathbb{S}^{d-1}).$$

- *The realization space of all inscribed polytopes combinatorially equivalent to it, modulo orthogonal transformations:*

$$(5) \quad \mathcal{R}_{ins}^{ort}(P) = \left\{ A \in (\mathbb{S}^{d-1})^n \mid \text{conv}(A) \simeq P \right\} / \text{O}(\mathbb{R}^d).$$

**2.3. Mnëv’s universality theorem.** A *primary basic semi-algebraic set* is a subset of  $\mathbb{R}^d$  defined by integer polynomial equations and inequalities

$$S = \left\{ \mathbf{x} \in \mathbb{R}^d \mid f_1(\mathbf{x}) = 0, \dots, f_k(\mathbf{x}) = 0, f_{k+1}(\mathbf{x}) > 0, \dots, f_r(\mathbf{x}) > 0 \right\}, \text{ where } f_i \in \mathbb{Z}[\mathbf{x}].$$

Realization spaces of polytopes and oriented matroids are examples of primary basic semi-algebraic sets. Mnëv’s Universality Theorem [Mnë88] is a reciprocal statement: every primary basic semi-algebraic set appears as the realization space of some oriented matroid/polytope up to *stable equivalence*, which implies homotopy equivalence (see [RG98]). We refer to [Mnë88][RG95][RG98] for its proof.

**Theorem 2.4** (Universality Theorem [Mnë88]). *For every primary basic semi-algebraic set  $S$  defined over  $\mathbb{Z}$  there is a rank 3 oriented matroid whose realization space is stably equivalent to  $S$ . If moreover  $S$  is open, then the oriented matroid may be chosen to be uniform.*

*Given any presentation of  $S$ , such an oriented matroid of size linear in the size of the presentation can be found in polynomial time. In particular, there is such a matroid whose size is linear in the sum of the arithmetic complexities of the polynomials.*

The *arithmetic complexity* of a polynomial  $f \in \mathbb{Z}[x]$  is, roughly speaking, the minimal number of operations  $+$  and  $\times$  needed to compute it from  $x$  and  $1$ , when we are allowed to reuse computations. For example,  $(x+1)^2 = (x+1)(x+1)$  can be computed with one addition and one multiplication. See [BCS97][Val79] for details.

The following statements are among the consequences of the Universality Theorem:

**Corollary 2.5.** *The realizability problem for oriented matroids of rank 3 is polynomially equivalent to the existential theory of the reals (ETR).*

**Corollary 2.6.** *For every finite field extension  $F/\mathbb{Q}$  of the rationals, there exists an oriented matroid of rank 3 that cannot be realized with coordinates in  $F$ .*

In other words, that “all algebraic numbers” are needed to coordinatize oriented matroids.

### 3. UNIVERSALITY FOR INSCRIBED POLYTOPES AND DELAUNAY SUBDIVISIONS.

In this section, we prove:

**Theorem 3.1.** *For every primary basic semi-algebraic set there is a Delaunay subdivision and an inscribed polytope whose realization space is homotopy equivalent to  $S$ .*

To pass from realization spaces of oriented matroids to those of polytopes, we use (as in [Mnë88]) Lawrence extensions. The resulting polytopes are always inscribable, as observed in [AZ14].

**3.1. Lawrence polytopes.** We recall some properties of polytopes constructed from Lawrence extensions.

**Definition 3.2** (cf. [RG98]). Let  $A$  be a  $d$ -dimensional point configuration and let  $a \in A$ . The *Lawrence extension* of  $A$  on  $a$  is the  $(d+1)$ -dimensional point configuration

$$\Lambda(A, a) := (A \setminus a) \cup \bar{a} \cup \underline{a},$$

where  $A$  is embedded in the hyperplane  $x_{d+1} = 0$  and the new points are  $\underline{a} := (a, 1)$  and  $\bar{a} := (a, 2)$ . Let  $B \subseteq A$ , the *Lawrence extension*  $\Lambda(A, B)$  is the point configuration obtained by Lawrence lifting the points of  $B$  one by one

$$\Lambda(A, B) := \Lambda(\Lambda(\dots \Lambda(\Lambda(A, b_1), b_2) \dots b_{k-1}), b_k),$$

where  $B = \{b_1, \dots, b_k\}$ .

The *Lawrence polytope* of a point configuration  $A$  is the polytope  $\Lambda(A) = \text{conv}(\Lambda(A, A))$ .

**Lemma 3.3** (cf. [Zie95, Theorems 6.26 and 6.27]). *For any point configuration  $A$ ,  $\Lambda(A, A)$  is in convex position and the Lawrence polytope  $\text{conv}(\Lambda(A, A))$  is rigid.*

**3.2. Partially inscribed point configurations.** Given an oriented matroid  $M$  and a subset of its elements  $E$ , we consider the set of realizations of  $M$  such that the points of  $E$  lie on the boundary of the unit ball  $\mathbb{B}^d$  and all the remaining points are outside. We use the homogenized version and consider such realizations up to orthogonal transformations fixing the hyperplane  $x_{d+1} = 0$ .

$$\mathcal{R}_{\text{om,ins}}(M, E) = \{V \in \mathbb{R}^{(d+1) \times n} \mid V \text{ realizes } M, \forall e \in E, A_e \in \partial \text{pos}(\text{hom}(\mathbb{B}^d)) \text{ and } \forall e \notin E, A_e \notin \text{pos}(\text{hom}(\mathbb{B}^d))\} / \text{O}(\mathbb{R}^d).$$

Notice that with this definition we are implicitly allowing points at infinity, and negative points, when we consider vectors that span rays not intersecting the homogenizing hyperplane.

The following lemma expands [AZ14, Proposition A.5.8] to make a statement about realization spaces.

**Lemma 3.4.** *For every planar point configuration  $A$ ,  $\mathcal{R}_{\text{om}}(A) \sim \mathcal{R}_{\text{ins}}(\Lambda(A))$ .*

*Proof.* Since Lawrence polytopes are rigid, and using Lemma 2.3, we have that  $\mathcal{R}_{\text{ins}}(\Lambda(A)) \sim \mathcal{R}_{\text{om,ins}}(\Lambda(A), \Lambda(A))$ . Therefore, we just need to prove that

$$(6) \quad \mathcal{R}_{\text{om}}(A) \sim \mathcal{R}_{\text{om,ins}}(\Lambda(A), \Lambda(A))$$

We prove first that for every subset  $B \subseteq A \subset \mathbb{R}^d$  and for every  $a \in A \setminus B$ ,

$$(7) \quad \mathcal{R}_{\text{om,ins}}(A, B) \sim \mathcal{R}_{\text{om,ins}}(\Lambda(A, a), B \cup \{\underline{a}, \bar{a}\}).$$

For every realization of  $\Lambda(A, a)$  one can recover a realization of  $A$  by intersecting the ray emanating at  $\bar{a}$  through  $\underline{a}$  with the linear hyperplane  $H$  spanned by the remaining points. For the moment, assume  $H$  is an equator of the unit sphere  $\mathbb{S}^{d-1}$ . In this case, it is clear we recover a realization of  $A$  with all the points of  $B$  on  $\mathbb{S}^{d-2}$  and all the remaining points outside  $\mathbb{B}^{d-1}$ . In general,  $H$  will not be an equator, but then there is a unique rescaling that sends  $H \cap \mathbb{S}^{d-1}$  to  $\mathbb{S}^{d-2}$ . This map is a well-defined projection

$$(8) \quad \mathcal{R}_{\text{om,ins}}(\Lambda(A, a), B \cup \{\bar{a}, \underline{a}\}) \longrightarrow \mathcal{R}_{\text{om,ins}}(A, B)$$

because every orthogonal transformation of  $\mathbb{R}^d$  induces an orthogonal transformation on  $H$ .

What's left is to establish that the fibers of this continuous map are non-empty and contractible. First, by reflection symmetry we may assume that  $\underline{a}$  and  $\bar{a}$  are in the positive half-space defined by  $H$ . With this, we then see that a point in the fiber is parameterized by the center of the sphere and the location of  $\bar{a}$ . This is the product of a line and a non-empty (spherically) convex subset of the sphere (points in the upper spherical cap visible from  $a$ ), and so contractible. Hence, (7) is established. Finally, we construct a continuous inverse by selecting  $\bar{a}$  to be the barycenter of the possible locations; since the fibers behave Hausdorff continuous on the pair, we are done.

To get to (6), we will prove that  $\mathcal{R}_{\text{om}}(A) \sim \mathcal{R}_{\text{om,ins}}(\Lambda(A, a), \{\underline{a}, \bar{a}\})$  for any  $a \in A$  (and then we only need to apply (7) to the remaining points). Here the fibers of the projection are the set of choices



for the sphere (the spheres touching the upper half-space not containing any point of  $A$ ) product with the choices for  $\bar{a}$  (again, a convex set). We can factor the projection map through the quotient  $\mathrm{GL}(\mathbb{R}^d)/\mathrm{O}(\mathbb{R}^d)$ .  $\square$

**3.3. Topological universality.** Now Theorem 3.1 follows directly from the combination of the Universality Theorem 2.4 with Lemma 3.4.

*Proof of Theorem 3.1.* By the Universality Theorem 2.4, for every primary basic semi-algebraic set  $S$  there is a point configuration whose realization space  $\mathcal{R}_{\mathrm{om}}(A)$  is homotopy equivalent to  $S$ . Now, by Lemma 3.4,  $\mathcal{R}_{\mathrm{om}}(A) \sim \mathcal{R}_{\mathrm{ins}}(\Lambda(A))$ . Finally, if we consider the Delaunay subdivision  $T$  consisting of a single cell combinatorially equivalent to  $\Lambda(A)$ , one can easily see that  $\mathcal{R}_{\mathrm{del}}(T) \sim \mathcal{R}_{\mathrm{ins}}(\Lambda(A))$ .  $\square$

**3.4. Algebraic universality.** Corollary 2.6 follows at once from the Universality Theorem 2.4 because of stable equivalence. Although the exact notion of stable equivalence does not hold in our situation, the statement analogous to Corollary 2.6 does.

**Corollary 3.5.** *For every finite field extension  $F/\mathbb{Q}$  of the rationals, there is a realizable Delaunay subdivision (equivalently, an inscribed polytope) that cannot be realized with coordinates in  $F$ .*

*Proof.* By Corollary 2.6, for every algebraic extension  $F$  of the rational numbers, there is a point configuration  $A$  that cannot be coordinatized in  $F$ . Now, by Lemma 3.4, the Lawrence polytope  $\Lambda(A)$  is inscribable. Any inscribed realization of  $\Lambda(A)$  encodes a realization of  $A$ , which can be obtained through a series of radial projections (see the proof of Lemma 3.4). Hence, if  $\Lambda(A)$  had a realization with coordinates in  $F$ , so would  $A$ .  $\square$

#### 4. UNIVERSALITY FOR INSCRIBED SIMPLICIAL POLYTOPES AND DELAUNAY TRIANGULATIONS

To obtain universality results for simplicial polytopes and triangulations, we cannot use Lawrence extensions, which produce configurations with a lot of non-simplicial faces. Instead, we will use *neighborly polytopes*, which are also rigid. This is possible, by a result of Kortenkamp, which implies that we can embed the oriented matroids of a planar point configurations inside the oriented matroid of a neighborly polytope.

**4.1. Stereographic projections.** The *stereographic projection*  $\phi : \mathbb{S}^d \setminus \mathbf{N} \subset \mathbb{R}^{d+1} \rightarrow \mathbb{R}^d$  is the map defined by

$$\phi(x_1, \dots, x_{d+1}) = \left( \frac{x_1}{1 - x_{d+1}}, \dots, \frac{x_d}{1 - x_{d+1}} \right),$$

where  $\mathbf{N}$  is the north pole of the unit sphere  $\mathbb{S}^d$ .

The stereographic projection and its inverse are classical tools to translate from Delaunay triangulations to inscribed polytopes, and vice versa [Bro79]. The following lemma explains how to relate realizations of the Delaunay triangulations and inscribed realizations of polytopes.

**Lemma 4.1.**  *$A = \{a_1, \dots, a_n\}$  be a configuration of  $n$  points in  $\mathbb{R}^d$ , and let  $\mathring{A} = \{\mathring{a}_1, \dots, \mathring{a}_n\}$  be its image under the inverse stereographic projection,  $\mathring{A} = \phi^{-1}(A)$ .*

*Then*

- (i)  $\mathring{a}_i$  is above (resp. on, below) the hyperplane spanned by  $\{\mathring{a}_{j_1}, \dots, \mathring{a}_{j_{d+1}}\}$  if and only if  $a_i$  is outside (resp. on, inside) the circumsphere spanned by  $\{a_{j_1}, \dots, a_{j_{d+1}}\}$ ; and
- (ii) for every hyperplane  $H \subset \mathbb{R}^d$ , there is a hyperplane  $\mathring{H} \subset \mathbb{R}^{d+1}$  with  $\mathbf{N} \in \mathring{H}$  such that  $\mathring{a}_i$  in  $\mathring{H}$  (resp.  $\mathring{H}^\pm$ ) if and only if  $a_i$  in  $H$  (resp.  $H^\pm$ ).

**Lemma 4.2.** *Let  $T$  be a  $d$ -dimensional polytopal subdivision with  $n$  vertices whose boundary is a  $d$ -simplex  $P$ , and let  $Q$  be polytopal complex (homeomorphic to a sphere) obtained by adding to  $T$  the cones with apex  $a_{n+1}$  over the faces of  $P$ . Then the stereographic projection from  $a_{n+1}$  induces a homeomorphism*

$$\mathcal{R}_{ins}(Q) \cong \mathcal{R}_{del}(T).$$

*Proof.* Let  $A = \{a_1, \dots, a_n, a_{n+1}\} \subset \mathbb{S}^d$  be an inscribed realization of  $Q$ . By a Möbius transformation, we can assume that the last point lies at the north pole,  $a_{n+1} = \mathbf{N}$ . Now, by Lemma 4.1, the Delaunay subdivision of the stereographic projection of the points  $a_i$ ,  $1 \leq i \leq n$ , coincides with  $T$ . Indeed, if  $S \subset A$  is the set of vertices of a facet  $F$  of  $\text{conv}(A \cup \mathbf{N})$  that does not contain  $\mathbf{N}$ , then  $S$  spans a supporting hyperplane that has all the remaining points above it (at the same side as  $\mathbf{N}$ ). According to Lemma 4.1(i), its stereographic projection  $S = \phi(S)$  spans an empty circumsphere, and hence is the set of vertices of a cell of the Delaunay subdivision of  $A$ . Additionally, by Lemma 4.1(ii) facets of  $\text{conv}(A \cup \mathbf{N})$  that contain  $\mathbf{N}$  are in bijection with facets of  $\text{conv}(A)$ , which by hypothesis is a simplex in any realization of  $T$ .

Moreover, every Möbius transformation of  $\mathbb{S}^d$  that fixes the north pole induces a similarity of  $\mathbb{R}^d$ . To conclude the proof, observe that every realization of  $T$  as Delaunay triangulation can be lifted with the inverse stereographic projection to a unique inscribed realization of  $Q$ .  $\square$

**Remark 4.3.** Notice that the exactly the same proof shows a bijection between realization spaces of inscribed polytopes and realization spaces of Delaunay subdivisions *with prescribed boundary*. Indeed, Lemma 4.1(ii) implies that the vertex figure of  $\mathbf{N}$  is combinatorially equivalent to the convex hull of the Delaunay triangulation. Since a general triangulation (as a simplicial complex) does not prescribe the convex hull of the realization, we have to focus only in those whose convex hull is a simplex.

**4.2. Lexicographic liftings.** A central tool for our construction are lexicographic liftings, which are a way to derive  $(d+1)$ -dimensional point configurations from  $d$ -dimensional point configurations.

**Definition 4.4.** A *lexicographic lifting* of a point configuration  $A = \{a_1, \dots, a_n\} \subset \mathbb{R}^d$  (with respect to the order induced by the labels) with a sign vector  $(s_1, \dots, s_n) \in \{+, -\}^n$  is a configuration  $\hat{A} = \{\hat{a}_1, \dots, \hat{a}_n, \hat{a}_{n+1}\}$  of  $n+1$  labeled points in  $\mathbb{R}^{d+1}$  such that:

- (i) for  $1 \leq i \leq d$ ,  $\hat{a}_i = (a_i, 0) \in \mathbb{R}^{d+1}$ ,
- (ii) for  $d+1 \leq i \leq n$ , the point  $\hat{a}_i$  lies in the half-line that starts at  $\hat{a}_{n+1}$  and goes through  $(a_i, 0)$ ,
- (iii) for  $d+1 \leq i \leq n$ , and for every hyperplane  $H$  spanned by  $d+1$  points of  $\{\hat{a}_1, \dots, \hat{a}_{i-1}\}$ , the points  $\hat{a}_{n+1}$  and  $\hat{a}_i$  lie at the same side of  $H$  when  $s_i = +$ , and at opposite sides if  $s_i = -$ .

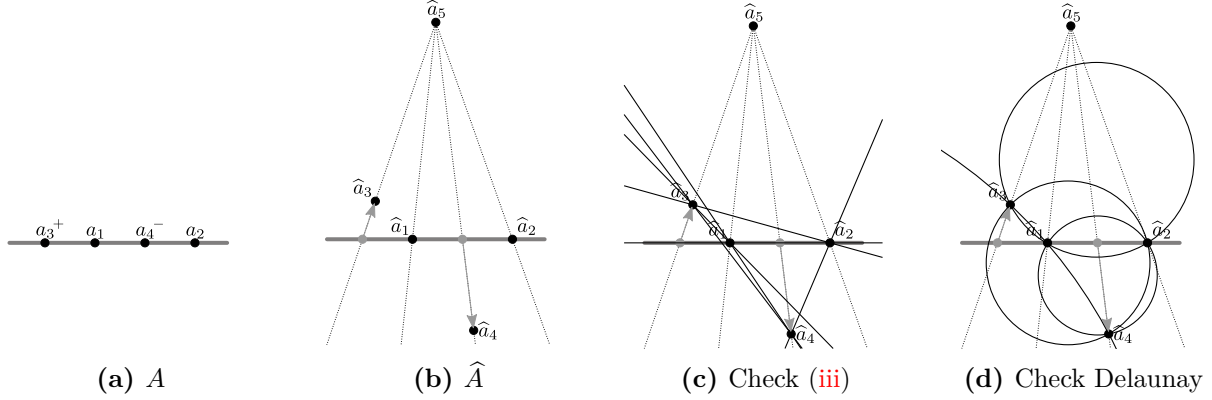
If  $s_i = +$  for every  $1 \leq i \leq n$ , the lexicographic lifting is called *positive*.

The proof of the following lemma is straightforward, since one can easily compute the chirotope of  $\hat{A}$  from that of  $A$  (compare [BLS<sup>+</sup>93, Chapter 7]).

**Lemma 4.5.** *The oriented matroid of a lexicographic lifting  $\hat{A}$  of  $A$  only depends on the oriented matroid of  $A$  and the sequence of signs.*

An alternative way to see this is to observe that lexicographic liftings are dual to lexicographic extensions (cf. [BLS<sup>+</sup>93, Section 7.2]).

**Remark 4.6.** A lexicographic lifting of  $A$  with signs  $s_i$  realizes the dual oriented matroid of a lexicographic extension of the Gale dual of  $A$  with signature  $[a_n^{-s_n}, \dots, a_{d+2}^{-s_{d+2}}]$ .



**Figure 4.1.** A Delaunay lexicographic lifting  $\hat{A} \subset \mathbb{R}^2$  of a configuration  $A \subset \mathbb{R}^1$ . In 4.1(c) one can check that (iii) is fulfilled, and in 4.1(d) that it is a Delaunay lexicographic lifting. This lifting is not positive.

We use lexicographic liftings because they preserve homotopy of realization spaces (although for our proof we only need the surjectivity of the map  $\mathcal{R}_{\text{om}}(\hat{A}) \rightarrow \mathcal{R}_{\text{om}}(A)$ ). This fact can be found in [BLS<sup>+</sup>93, Lemma 8.2.1 and Proposition 8.2.2].

**Lemma 4.7.** *For any lexicographic lifting  $\hat{A}$  of  $A$ ,  $\mathcal{R}_{\text{om}}(\hat{A})$  is homotopy equivalent to  $\mathcal{R}_{\text{om}}(A)$ .*

*Proof (sketch).* Any vector configuration  $\hat{V} = \{\hat{v}_1, \dots, \hat{v}_{n+1}\}$  with the same oriented matroid as of  $\text{hom}(\hat{A})$  can be mapped to a configuration  $V = \{v_1, \dots, v_n\}$  that realizes  $\text{hom}(A)$ , just by taking  $v_i$  to be the orthogonal projection of  $\hat{v}_i$  onto the hyperplane orthogonal to  $\hat{v}_{n+1}$ . This defines a continuous map from  $\mathcal{R}_{\text{om}}(\hat{A})$  to  $\mathcal{R}_{\text{om}}(A)$ , which is easily seen to be surjective (compare Lemma 4.9).

To see that this is indeed a homotopy equivalence, we can check that the fibers of this projection are balls. Indeed, once the position of  $\hat{v}_{n+1}, \hat{v}_n, \dots, \hat{v}_{i+1}$  is fixed, the set of valid positions of  $\hat{v}_i$  is a convex subset of the line that goes through  $\hat{v}_{n+1}$  and  $v_i$ .  $\square$

To control their Delaunay triangulations, we use a particular family of lexicographic liftings (see also [GP13] and [Sei85]).

**Definition 4.8.** A *Delaunay lexicographic lifting* of  $A$  is a lexicographic lifting  $\hat{A}$  such that for each  $d+2 < i \leq n+1$ ,  $\hat{a}_i$  is not contained in any of the circumspheres of any simplex spanned by  $d+2$  points of  $\{\hat{a}_1, \dots, \hat{a}_{i-1}\}$ .

**Lemma 4.9.** *For any point configuration  $A$  and any  $s \in \{+, -\}^n$ , there is Delaunay lexicographic lifting.*

*Proof.* To construct one, just replace  $a_i$  by  $\hat{a}_i = (a_i, h_i) \in \mathbb{R}^{d+1}$  for some  $h_i$  large enough so that  $\hat{a}_i$  is above or below every hyperplane spanned by  $\{\hat{a}_1, \dots, \hat{a}_{i-1}\}$  and outside any of the circumspheres spanned by  $\{\hat{a}_1, \dots, \hat{a}_{i-1}\}$ . Finally, set  $\hat{a}_{n+1}$  to be the “point at infinity”  $\hat{a}_{n+1} = (0, +\infty)$  and apply a projective transformation that preserves the hyperplane spanned by  $\{a_1, \dots, a_d\}$  and sends  $\hat{a}_{n+1}$  to  $(0, h_{n+1})$  for some large  $h_{n+1} > 0$ . (This is possible, because the  $\hat{a}_i$  are chosen so that the empty sphere condition will hold after a small perturbation.)  $\square$

We end with the following straightforward consequence of Lemma 4.1.

**Corollary 4.10.** *Let  $A = \{a_1, \dots, a_n\}$  be a configuration of  $n$  labeled points in general position in  $\mathbb{R}^d$  and let  $\hat{A}$  be a Delaunay lexicographic lifting of  $A$ . Then  $\phi^{-1}(\hat{A}) \cup \mathbf{N}$  is a positive lexicographic lifting of  $\hat{A}$  (with respect to the same order) inscribed on  $\mathbb{S}^{d+1}$ .*

*Proof.* The condition that  $\widehat{A}$  is a Delaunay lexicographic lifting implies that  $a_j$  is outside every circumsphere spanned by points in  $\{a_1, \dots, a_{j-1}\}$ , which by Lemma 4.1 implies that  $a_j$  is above (i.e. at the same side as  $\mathbf{N}$ ) every hyperplane spanned by points in  $\{a_1, \dots, a_{j-1}\}$ .  $\square$

**4.3. Neighborly oriented matroids.** A crucial property of even-dimensional neighborly configurations is that their oriented matroids are *rigid*.

**Theorem 4.11** ([Stu88, Theorem 4.2][She82]). *If  $A$  is an even-dimensional neighborly point configuration, then the oriented matroid of  $A$  is rigid, i.e.  $\mathcal{R}_{om}(A) = \mathcal{R}_{pol}(A)$ .*

Kortenkamp [Kor97] found a way to use lexicographic liftings to construct neighborly point configurations.

**Theorem 4.12** ([Kor97, Theorem 1.2]). *For any point configuration  $A$  with  $d + 4$  points in general position in  $\mathbb{R}^d$  there is an even-dimensional neighborly configuration  $\widehat{A}$  of  $2d + 8$  points in  $\mathbb{R}^{2d+4}$  obtained from  $A$  by a sequence of lexicographic liftings.*

Finally, the following result can be found in [Pad13] (compare also [GP13]), where it is used to construct many neighborly polytopes.

**Theorem 4.13** ([Pad13, Theorem 4.2]). *Let  $A$  be a neighborly point configuration in general position, let  $\widehat{A}$  be a lexicographic lifting of  $A$  and let  $\widehat{\widehat{A}}$  be a positive lexicographic lifting of  $\widehat{A}$  (with respect to the same order). Then  $\widehat{\widehat{A}}$  is neighborly.*

**4.4. The construction.** Here is the main technical result of this section.

**Lemma 4.14.** *For every configuration  $A$  of  $n$  points in general position in  $\mathbb{R}^{n-4}$  there exists an inscribed neighborly polytope  $P$  with  $2n + 2$  vertices in  $\mathbb{R}^{2n-2}$  and sets  $X$  and  $Y$ , homotopy equivalent to  $\mathcal{R}_{ins}(P)$  and  $\mathcal{R}_{om}(A)$  respectively, such that  $Y$  is a retract of  $X$ .*

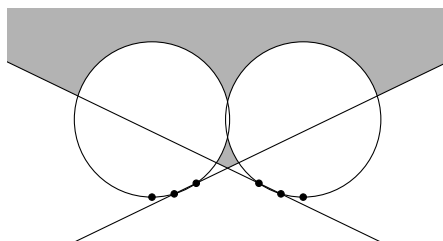
*Proof.* For convenience, set  $d = n - 4$ . Since  $A$  is a  $d$ -dimensional configuration of  $d + 4$  points, we can apply the sequence of lexicographic liftings of Theorem 4.12 to obtain a neighborly configuration  $A_2$  of  $2n$  points in general position in  $\mathbb{R}^{2n-4}$ . This configuration is obtained by lexicographic liftings and hence the corresponding realization spaces are homotopy equivalent,  $\mathcal{R}_{om}(A) \sim \mathcal{R}_{om}(A_2)$ , by Lemma 4.7.

Now, we can apply a lexicographic lifting and a positive lexicographic lifting successively to obtain  $A_3 = \widehat{\widehat{A_2}}$ , which is a configuration of  $N = 2n + 2$  points in general position in  $\mathbb{R}^D$ , where  $D = 2n - 2$ . The convex hull of  $A_3$  is a neighborly polytope  $P$  by Theorem 4.13. We will build a continuous surjection from  $\mathcal{R}_{ins}(P) \times \mathbb{R}_{>0}^{N-1}$  onto  $\mathcal{R}_{om}(A_2)$ .

Let  $B \subset \mathbb{R}^{N \times D}$  be an inscribed realization of  $P$ , which is even-dimensional and neighborly. By Theorem 4.11, its oriented matroid is rigid, and hence the matroid of the vertices of  $P$  coincides with the matroid of  $A_3$ . Therefore, the stereographic projection  $\phi(B)$  of  $B$  from  $a_N$  is always a realization of  $\widehat{A_2}$ . Consider then the map  $\varphi : \mathcal{R}_{ins}(P) \times \mathbb{R}_{>0}^{N-1} \rightarrow \mathcal{R}_{om}(\widehat{A_2})$  that maps  $(B, \lambda)$  onto the configuration of vectors  $\{(\lambda_i \phi(B_i), \lambda_i)\}_{1 \leq i \leq N-1} \in \mathcal{R}_{om}(\widehat{A_2}) \subseteq \mathbb{R}^{(N-1) \times D}$ . The map is well defined, because Möbius transformations of  $B$  induce similarities of  $\phi(B)$ , which are affine transformations and hence induce linear transformations on  $\varphi(B, \lambda)$ .

Now we can use the projection map  $\psi : \mathcal{R}_{om}(\widehat{A_3}) \rightarrow \mathcal{R}_{om}(A_3)$  of Lemma 4.7 to obtain a realization of  $A_3$ . From Corollary 4.10 we deduce that the composition map  $\psi \circ \varphi : \mathcal{R}_{ins}(P) \times \mathbb{R}_{>0}^{N-1} \rightarrow \mathcal{R}_{om}(A_3)$  is surjective. To conclude that  $\psi \circ \varphi$  is a retraction, we have to exhibit a continuous inverse injection. But the construction of Corollary 4.10 can easily be performed in a continuous way. For example, one

can use Hadamard's determinant inequalities to find continuous heights that fulfill the constraints of Delaunay lexicographic liftings.  $\square$



**Figure 4.2.** Any point in the shaded area gives rise to the same Delaunay triangulation.

The reason why we cannot strengthen the statement to homotopy equivalence between  $\mathcal{R}_{\text{ins}}(P)$  and  $\mathcal{R}_{\text{om}}(A)$ , is that we do not understand the fibers of the map  $\mathcal{R}_{\text{ins}}(P) \rightarrow \mathcal{R}_{\text{om}}(A_3)$ . We can prove that they are non-empty with Corollary 4.10 but we cannot control their topology. (Figure 4.2 shows an example of how disconnected fibers might arise.)

**Theorem 4.15.** *For every open primary basic semi-algebraic set  $S$  there is a (neighborly) Delaunay triangulation and an inscribed simplicial (neighborly) polytope such that  $S$  is a retract of their realization spaces, up to homotopy equivalence.*

*Proof.* A straightforward consequence of the Universality Theorem 2.4 is that realization spaces of oriented matroids of configurations of  $d + 4$  points in  $\mathbb{R}^d$  exhibit universality. In particular, for every open primary basic semi-algebraic set  $S$  there is a configuration  $A$  of  $d + 4$  points in general position in  $\mathbb{R}^d$  whose realization space is homotopy equivalent to  $S$ . The proof is direct using oriented matroid duality (see [BLS<sup>+</sup>93, Chapter 8]) after reorienting some elements (compare [Zie95, Corollary 6.16]).

Hence, by Lemma 4.14, there is an inscribed simplicial neighborly  $d$ -polytope  $P$  whose realization space admits a continuous surjection onto a set homotopy equivalent to  $S$ .

For the claim concerning Delaunay triangulations, we consider the polytope  $P'$  obtained by stacking a vertex on the facet  $F = \{a_1, \dots, a_d\}$  of  $P$ . (The face lattice of  $P'$  coincides with that of  $P$ , except that  $F$  is replaced with its stellar subdivision.)

We claim that every realization of  $A$  can be lifted to an inscribed realization of  $P'$  (and by construction, every realization of  $P'$  can be projected to a realization of  $A$ ). Indeed, to the configuration  $A_2$  of Lemma 4.14, add a point  $a_0$  in the relative interior of  $\text{conv}(a_1, \dots, a_{d-1})$  and then apply a positive Delaunay lexicographic lifting with order  $a_1, \dots, a_{d-1}, a_0, a_d, a_{d+1}, \dots$ . The Delaunay triangulation of this configuration clearly contains the stellar subdivision of the simplex  $\{a_1, \dots, a_d\}$ .

An application of Lemma 4.2 then concludes the proof.  $\square$

**4.5. Complexity.** A closer look into the proof of Lemma 4.14 shows that all the operations that we use are at the oriented matroid level (i.e., can be also applied to non-realizable matroids) and take only polynomial time. Therefore, for each rank 3 oriented matroid  $M$  we can construct a (combinatorial) Delaunay triangulation that is realizable *if and only if*  $M$  is. An important consequence of the Universality Theorem is Corollary 2.5, which states that realizability of rank 3 oriented matroids is polynomially equivalent to ETR [Mnë88][Sho91]. Lemma 4.14 implies that realizability of Delaunay triangulations is equally hard.

**Corollary 4.16.** *The realizability problem for Delaunay triangulations and simplicial inscribed polytopes is polynomially equivalent to the existential theory of the reals (ETR).*

Another consequence of the universality theorem for Delaunay triangulations is that realization spaces can have an exponential number of connected components.

**Corollary 4.17.** *For every  $m \geq 1$  there exist configurations of  $O(m)$  points in general position in  $\mathbb{R}^{O(m)}$  whose realization spaces as Delaunay triangulations have at least  $2^m$  connected components.*

*Proof.* Consider the polynomial  $f_m(x)$  obtained recursively as follows:

$$f_0(x) = x^2 - 2, \quad f_{k+1}(x) = f_k(f_0(x)).$$

That is,  $f_1(x) = (x^2 - 2)^2 - 2$ ,  $f_2(x) = ((x^2 - 2)^2 - 2)^2 - 2$ ,  $f_3(x) = (((x^2 - 2)^2 - 2)^2 - 2)^2 - 2$ , and so on. It is not hard to check that  $f_m(x)$  has  $2^{m+1}$  distinct simple real roots and that its arithmetic complexity is  $O(m)$ . The semi-algebraic set of points fulfilling  $f_m(x) > 0$  has at least  $2^m$  connected components.

Our claim now follows by the Universality Theorem 2.4 and Lemma 4.14.  $\square$

As a final remark in this section, we provide our smallest example of a Delaunay triangulation with disconnected realization space. It can be constructed by applying Lemma 4.14, together with the stacking technique of the proof Theorem 4.15, to the uniform rank 3 oriented matroid with 14 elements found by Suvorov in 1988 [Suv88] (see also [BLS<sup>+</sup>93, Chapter 8]), which has a disconnected realization space.

**Corollary 4.18.** *There is a 25-dimensional configuration of 30 points whose Delaunay triangulation has a disconnected realization space.*

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